December 2, 2008

WEAK AND STRONG TAYLOR METHODS FOR NUMERICAL SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. We apply results of Malliavin-Thalmaier-Watanabe for strong and weak Taylor expansions of solutions of perturbed stochastic differential equations (SDEs). In particular, we work out weight expressions for the Taylor coefficients of the expansion. The results are applied to LIBOR market models in order to deal with the typical stochastic drift and with stochastic volatility. In contrast to other accurate methods like numerical schemes for the full SDE, we obtain easily tractable expressions for accurate pricing. In particular, we present an easily tractable alternative to "freezing the drift" in LIBOR market models, which has an accuracy similar to the full numerical scheme. Numerical examples underline the results.

1. Introduction and Setting

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying an N-dimensional Brownian motion $(W_t)_{t\geq 0}$ with a $d \times d$ correlation matrix. We consider smooth curves $F_\epsilon : \mathbb{R} \to L^2(\Omega; \mathbb{R}^N)$ of random variables, where $\epsilon \in \mathbb{R}$ is a parameter. We apply Taylor theorems to obtain strong approximations of the curve F_ϵ at $\epsilon = 0$ and we apply partial integration on Wiener space to obtain weak approximations of the law of F_ϵ for small values of ϵ.

We choose the notion *Taylor expansion* instead of asymptotic expansion in order to point out that the strong method is indeed a classical Taylor expansion with usual conditions for convergence. The weak method represents a truncated converging power series in the parameter ϵ if – for instance – the payoff $f : \mathbb{R}^N \to \mathbb{R}$ stems from a real analytic function and some distributional properties are satisfied.

2. Weak and strong Taylor methods - Structure Theorems

We introduce in this section two concepts of approximation. Consider a curve $\epsilon \mapsto F_\epsilon$, where $\epsilon \in \mathbb{R}$ and $F_\epsilon \in L^2(\Omega; \mathbb{R}^N)$.

Definition 1. *A* ***strong Taylor approximation*** *of order $n \geq 0$ is a (truncated) power series*

$$\mathbf{T}^n_\epsilon(F_\epsilon) := \sum_{i=0}^{n} \frac{\epsilon^i}{i!} \frac{\partial^i}{\partial \epsilon^i}\Big|_{\epsilon=0} F_\epsilon, \tag{2.1}$$

such that

$$\mathbb{E}\big(|F_\epsilon - \mathbf{T}^n_\epsilon(F_\epsilon)|\big) = o(\epsilon^n), \tag{2.2}$$

Financial support from the Austrian Science Fund (FWF) under grant P 15889 and the START-prize-grant Y328-N13 is gratefully acknowledged. Furthermore this work was financially supported by the Christian Doppler Research Association (CDG). The authors gratefully acknowledge a fruitful collaboration and continued support by Bank Austria and the Austrian Federal Financing Agency (ÖBFA) through CDG.

holds true as $\epsilon \to 0$*.*

Remark 1. *In our setting a strong Taylor approximation of any order* $n \geq 0$ *of the curve* F_ϵ *can always be obtained, see for instance* [KM97].

Let $f : \mathbb{R}^N \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant K, then we obtain

$$\mathbb{E}\big(|f(F_\epsilon) - f\big(\mathbf{T}^n_\epsilon(F_\epsilon)\big)|\big) \leq K\mathbb{E}\big(\|F_\epsilon - \mathbf{T}^n_\epsilon(F_\epsilon)\|\big) = Ko(\epsilon^n). \tag{2.3}$$

Equation (2.3) does not hold anymore if f is not globally Lipschitz continuous. In particular, we observe the dependence of the right hand side on the Lipschitz constant K. Hence, truncating an a-priori known Taylor expansion leads to an error term, which contains the Lipschitz constant and is therefore not useful for non-Lipschitz claims. The weak method navigates around this feature by partial integration.

Definition 2. *A* ***weak Taylor approximation*** *of order* $n \geq 0$ *is a power series for each bounded, measurable* $f : \mathbb{R}^N \to \mathbb{R}$*,*

$$\mathbf{W}^n_\epsilon(f, F_\epsilon) := \sum_{i=0}^{n} \frac{\epsilon^i}{i!} \mathbb{E}(f(F_0)\pi_i),$$

where $\pi_i \in L^1(\Omega)$ *denote real valued, integrable random variables, such that*

$$|\mathbb{E}\big(f(F_\epsilon)\big) - \mathbf{W}^n_\epsilon(f, F_\epsilon)| = o(\epsilon^n).$$

Remark 2. *The weights* π_i *for* $i \geq 1$ *are called* Malliavin weights.

Remark 3. *If the law of* F_ϵ *is real analytic at* $\epsilon = 0$ *in the weak sense, i.e. if there exist (signed) measures* μ_i *such that for all bounded, measurable* $f : \mathbb{R}^N \to \mathbb{R}$ *the following series converges and the equality*

$$\mathbb{E}\big(f(F_\epsilon)\big) = \sum_{i \geq 0} \frac{\epsilon^i}{i!} \int_{\mathbb{R}^N} f(x)\mu_i(dx),$$

holds true, precisely then we do have a converging weak Taylor expansion. We aim for constructing stochastic representations of the following type, for $i \geq 0$*:*

$$\int_{\mathbb{R}^N} f(x)\mu_i(dx) = \mathbb{E}(f(F_0)\pi_i).$$

For the definition of the weak Taylor approximation to make sense, existence of the Malliavin weights has to hold. The following theorem can be found in a slightly different version in [MT06] and goes back to S. Watanabe. For the definition and notion of $\mathcal{D}^\infty(\mathbb{R}^N)$ see [Mal97] or [Nua06].

Theorem 1. *Let* $F_\epsilon : \mathbb{R} \to \mathcal{D}^\infty(\mathbb{R}^N)$ *be smooth and assume that the Malliavin covariance matrix* $\gamma(F_\epsilon)$ *is invertible with p-integrable inverse for every* $p \geq 1$ *around* $\epsilon = 0$ *(i.e. on an open interval containing* $\epsilon = 0$*). Then there is a weak Taylor approximation of any order* $n \geq 0$ *and there are explicit formulas for the weights* π_i*. If we only know that the Malliavin covariance matrix* $\gamma(F_0)$ *is invertible with p-integrable inverse, then we can also calculate the Malliavin weights, since they depend only on* $\gamma(F_0)$*.*

Proof. Fix $n \geq 0$ and take a smooth test function $f : \mathbb{R}^N \to \mathbb{R}$ and assume that $\gamma^{-1}(F_\epsilon)$ exists as a smooth curve in $\mathcal{D}^\infty$ on a open ϵ-interval containing $\epsilon = 0$. By standard arguments we can prove the following formula

$$\frac{d}{d\epsilon}\mathbb{E}\big(f(F_\epsilon)\big) = \mathbb{E}\Bigg(f(F_\epsilon)\delta\Big(s \mapsto (D_s F_\epsilon)^{\mathrm{T}}\gamma^{-1}(F_\epsilon)\frac{d}{d\epsilon}F_\epsilon\Big)\Bigg).$$

More precisely, by the integration by parts [Nua06, Definition 1.3.1-(1.42)], the chain rule [Nua06, Proposition 1.2.3] and the definition of the Malliavin covariance matrix, [Nua06, page 92], we obtain from the right hand side the desired left hand side.

Notice that the ϵ-dependence of the Skorohod integral is smooth due to basic properties of $\mathcal{D}^\infty$. Hence, we can calculate higher derivatives of the left hand side by iterating the above procedure and differentiating the Skorohod integral. We denote

$$\pi_1 := \delta\Big(s \mapsto (D_s F_\epsilon)^{\mathrm{T}}\gamma^{-1}(F_\epsilon)\frac{d}{d\epsilon}F_\epsilon\Big). \tag{2.4}$$

We write then, pars pro toto, the formula for the second derivative

$$\begin{aligned}
\frac{d^2}{d\epsilon^2}\mathbb{E}(f(F_\epsilon)) = {}& \mathbb{E}\Bigg(f(F_\epsilon)\delta\Big(s \mapsto \pi_1 (D_s F_\epsilon)^{\mathrm{T}}\gamma^{-1}(F_\epsilon)\frac{dF_\epsilon}{d\epsilon}\Big)\Bigg)+\\
&+\mathbb{E}\Bigg(f(F_\epsilon)\delta\Big(s \mapsto \Big(D_s \frac{dF_\epsilon}{d\epsilon}\Big)^{\mathrm{T}}\gamma^{-1}(F_\epsilon)\frac{dF_\epsilon}{d\epsilon}\Big)\Bigg)-\\
&-\mathbb{E}\Bigg(f(F_\epsilon)\delta\Big(s \mapsto (D_s F_\epsilon)^{\mathrm{T}}\gamma^{-1}(F_\epsilon)\frac{d\gamma(F_\epsilon)}{d\epsilon}\gamma^{-1}(F_\epsilon)\frac{dF_\epsilon}{d\epsilon}\Big)\Bigg)+\\
&+\mathbb{E}\Bigg(f(F_\epsilon)\delta\Big(s \mapsto (D_s F_\epsilon)^{\mathrm{T}}\gamma^{-1}(F_\epsilon)\frac{d^2 F_\epsilon}{d\epsilon^2}\Big)\Bigg).
\end{aligned}$$

This formula makes perfect sense at $\epsilon = 0$ and – by induction – we see that we can perform this step for any derivative. The general, recursive result is the following:

$$\begin{aligned}
&a_s := (D_s F_\epsilon)^{\mathrm{T}}\gamma^{-1}(F_\epsilon)\frac{dF_\epsilon}{d\epsilon} \text{ for } 0 \leq s \leq T,\\
&\pi_n := \delta(s \mapsto a_s \pi_{n-1}) + \frac{d}{d\epsilon}\pi_{n-1},\\
&\pi_0 := 1.
\end{aligned}$$

Here we understand the weights π_n as ϵ-dependent, whereas in the final formulas we put $\epsilon = 0$. This proves the result for smooth test functions f and under the assumption that the Malliavin covariance matrix is invertible around $\epsilon = 0$. If we approximate a bounded, measurable function f by smooth test functions we obtain the desired assertion by standard arguments, since the weights are integrable. □

Remark 4. *By Taylor's theorem and the Faà-di-Bruno-formula we obtain*

$$\frac{d^n f(F_\epsilon)}{d\epsilon^n} = \sum_{|\alpha| \leq n} f^{(\alpha)}(F_\epsilon)p_\alpha,$$

where p_α is a well-defined polynomial in derivatives of the curve $\epsilon \mapsto F_\epsilon$, for a multi-index α. Since $\mathcal{D}^\infty$ is an algebra, see [Mal97], *the above expression lies in*

$L^p(\Omega)$ for each $p \geq 0$. The previous result provides a representation of the partial integration result for

$$\mathbb{E}(\sum_{|\alpha|\leq n} f^{(\alpha)}(F_\epsilon)p_\alpha) = E(f(F_\epsilon)\pi_n).$$

The structure of the weights is seen from above. The result can be considered as a dual version of the Faà-di-Bruno-formula. However, the structure of this dual formula is much simpler.

We provide an example to demonstrate the strong and weak method of approximation. The method works in order to replace time-consuming iteration schemes, like the Euler-scheme, by simulations of "simple" Itô integrals.

Example 1. *We deal with a generic, real-valued random variable over a one-dimensional Gaussian space, see* [Nua06], *i.e.*

$$F_\epsilon = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} F^i,$$

where the F^i lie in the $(i+1)^{st}$ Wiener chaos $\mathcal{H}_{i+1}(\Omega)$ (one can think of a Hermite expansion for instance) and the sum is understood in the L^2-sense. From the strong expansion we obtain immediately – for a given Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ – that

$$|\mathbb{E}\big(f(F_\epsilon)\big) - \mathbb{E}\big(f(F^0 + \epsilon F^1)\big)| \leq K o(\epsilon),$$

as $\epsilon \to 0$, where K denotes the Lipschitz constant of f. This simple approximation can be sometimes quite useful.

We assume now that $F^0 = \int_0^\infty h(s)dW_s$ has non-vanishing variance in order to calculate the weights, which do depend only on $\gamma(F_0)$. The strong Taylor approximation is given by definition, the weak Taylor expansion can be constructed by the previous recursive formulas and the specifications

$$\begin{aligned} D_s F^0 &= h(s), \\ \gamma(F^0) &= \int_0^\infty h(s)^2 ds, \\ a_s &= \frac{h(s)}{\int_0^\infty h(s)^2 ds}. \end{aligned}$$

In order to obtain a first-order approximation for bounded, measurable random variables we therefore have to calculate

$$\mathbb{E}\big(f(F^0)\big) + \epsilon\mathbb{E}\big(f(F^0)\pi_1\big),$$

where

$$\pi_1 = \delta\big(s \mapsto a_s F^1\big).$$

This amounts to an integration of f times a polynomial with respect to a Gaussian density, since:

$$\mathbb{E}\big(f(F^0)\pi_1\big) = \mathbb{E}\big(f(F^0)F^1 \int_0^\infty a_s dW_s\big) - \int_0^\infty \mathbb{E}\big(f(F^0)D_s F^1 a_s\big)ds.$$

Notice that the strong approximation does not yield such a result for bounded, measurable random variables. *Notice also that in the given case the approximation can be calculated in a deterministic way, since we deal with Gaussian integrations.*

The second-order weak Taylor approximation is given by

$$\mathbb{E}\big(f(F^0)\big) + \epsilon\mathbb{E}\big(f(F^0)\pi_1\big) + \epsilon^2\mathbb{E}\big(f(F^0)\pi_2\big),$$

where

$$\pi_2 = \delta\big(s \mapsto \pi_1 a_s F^1\big) + \delta\big(s \mapsto a_s F^2\big).$$

3. Applications from Financial Mathematics

For applications we want to deal with strong and weak Taylor approximations of a given curve of random variables. We are particulary interested in cases, where the first derivative $\frac{dF_\epsilon}{d\epsilon}|_{\epsilon=0}$ is of simple form or – even more important – where the Malliavin covariance matrix $\gamma(F_0)$ is of simple form. In these cases it is easy to obtain first or second order approximations of the respective quantities in the weak or strong sense.

In what follows, first we will present one of the most applied interest rate models, namely the LIBOR market model (LMM). Then, we will introduce the commonly used technique of *freezing the drift*. We will show how to embed the"freezing the drift" technique into our framework of Taylor approximations. We understand freezing the drift as a strong Taylor approximation of order zero in the drift term of the LIBOR SDE. Our goal is to put this technique into a method, where we can in particular improve the order of approximation. We will finally extend the assumption of log normality and develop a stochastic volatility LMM, where we will show how to obtain tractable option prices via our weak Taylor approximations.

3.1. **The LIBOR Market Model.** We apply our concepts to the LMM, initially constructed by [BGM97], [MSS97] and [Jam97]. Let T denote a strictly positive fixed time horizon and $(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_t)_{0\le t\le T})$ be a complete probability space, supporting an N-dimensional Brownian motion $W_t = (W_t^1, ...W_t^N)_{0\le t\le T}$. The factors are correlated with $dW_t^i\ dW_t^j = \rho_{ij}dt$. Let $0 = T_0 < T_1 < T_2 < \ldots < T_N < T_{N+1} =: T$ be a discrete tenor structure and $\alpha := T_{i+1} - T_i$ the accrual factor for the time period $[T_i, T_{i+1}]$, $i = 0, \ldots, N$. Let $P(t, T_i)$ denote the value at time t of a zero coupon bond with maturity $T_i \in [0, T]$. The measure $\mathbb{P}$ is the terminal forward measure, which corresponds to taking the final bond $P(t, T)$ as numéraire. The forward LIBOR rate $L_t^i := L_t(T_i, T_{i+1})$ at time $t \le T_i$ for the period $[T_i, T_{i+1}]$ is given by:

$$L_t^i = L_t(T_i, T_{i+1}) = \frac{1}{\alpha}\left(\frac{P(t, T_i)}{P(t, T_{i+1})} - 1\right).$$

We assume that for any maturity T_i there exists a bounded, continuous, deterministic function $\sigma^i(t) : [0, T_i] \to \mathbb{R}$, which represents the volatility of the LIBOR L_t^i, $i = 1, ..., N$. The log normal LIBOR market model can be expressed under the measure $\mathbb{P}$ as:

$$dL_t^i = \sigma^i(t)L_t^i\Big(-\sum_{j=i+1}^{N}\frac{\alpha L_t^j\sigma^j(t)}{1+\alpha L_t^j}\rho_{ij}\Big)dt + \sigma^i(t)L_t^i dW_t^i, i = 1, ..., N. \tag{3.1}$$

3.2. **Freezing the Drift.** The dynamics of forward LIBORs for $i = 1, ..., N-1$ depend on the stochastic drift term $\frac{\alpha L_t^j}{1+\alpha L_t^j}$, $i \le j \le N$, which is determined by LIBOR rates with longer maturities. This random drift prohibits analytic tractability when pricing products that depend on more that one LIBOR rate, since there is no unifying measure under which all LIBOR rates are simultaneously log normal.

In addition, it encumbers the numerical implementation of the model. Common practice is to approximate this term by its starting value $\frac{\alpha L_0^j}{1+\alpha L_0^j}$ or as it is widely referred to as *freezing the drift*, i.e.

$$\frac{\alpha L_t^j}{1+\alpha L_t^j} \approx \frac{\alpha L_0^j}{1+\alpha L_0^j}.$$

It was first implemented in the original paper [BGM97] for the pricing of swaptions based on the LMM. [BW00] and [Sch02] argue that freezing the drift is justified due to the fact that this term has small variance. However, by freezing the drift there is a difference in option prices with the real and the frozen drift. It has not been examined how big the error is or for which assets it works well or not. Our aim is to investigate such a phenomenon and improve the performance by providing with correction terms of order one.

3.3. **Correcting the Frozen Drift.** The purpose of this section is to embed the well-known and often applied technique of *freezing the drift* into the strong and weak Taylor approximations, in order to develop a method to improve the order of accuracy. Specifically for the strong Taylor approximation, the method works well, since we always deal with a globally Lipschitz drift term $x \mapsto \frac{\alpha x_+}{1+\alpha x_+}$ with small Lipschitz constant α.

Remark 5. *As it will be clear later, the strong Taylor correction method can be accommodated with any extension of the log normal LMM, for example with the Lévy LIBOR model by Eberlein and Özkan* [EÖ05].

3.3.1. *Strong Taylor Approximation.* We first state a useful lemma, asserting that we can indeed freeze the drift under special model formulation and choice parameters.

Lemma 1. *Let $\epsilon_1 \in \mathbb{R}$ and consider for $i = 1, \dots, N$ the following stochastic differential equation:*

$$dX_t^{(i,\epsilon_1)} = \epsilon_1 \left(\sigma^i(t) X_t^{(i,\epsilon_1)} \Big(- \sum_{j=i+1}^{N} \frac{\alpha X_t^{(j,\epsilon_1)} \sigma^j(t)}{1+\alpha X_t^{(j,\epsilon_1)}} \rho_{ij} dt + dW_t^i \Big) \right), \tag{3.2}$$

defined on the complete probability space $(\Omega, \mathcal{F}_T, \mathbb{P}, (\mathcal{F}_t)_{0\le t\le T})$ where W_t is an N-dimensional Brownian motion under the measure $\mathbb{P}$ with $dW_t^i dW_t^j = \rho_{ij} dt$. Then the first-order strong Taylor approximation for $X_t^{(i,\epsilon_1)}$ is given by:

$$\mathbf{T}^1_{\epsilon_1}(X_t^{(i,\epsilon_1)}) = X_t^{(i,0)} + \epsilon_1 \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} X_t^{(i,\epsilon_1)}. \tag{3.3}$$

Proof. By (1) we obtain for $n=1$:

$$\mathbf{T}^1_{\epsilon_1}(X_t^{(i,\epsilon_1)}) \simeq X_t^{(i,\epsilon_1)} = X_0^{(i,0)} + \epsilon_1 Y_t^i + o(\epsilon_1),$$

since $X_t^{(i,0)} = X_0^{(i,0)}$ and where $Y_t^i := \frac{\partial}{\partial \epsilon_1}|_{\epsilon_1=0} X_t^{(i,\epsilon_1)}$ is the first-order correction term. By differentiating (3.2) with respect to ϵ_1, we calculate:

$$d\Big(\frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} X_t^{(i,\epsilon_1)} \Big) = \sigma^i(t) X_0^{(i,0)} \Big(- \sum_{j=i+1}^{N} \frac{\alpha X_0^{(j,0)} \sigma^j(t)}{1+\alpha X_0^{(j,0)}} \rho_{ij} dt + dW_t^i \Big),$$

and derive Y_t^i as the solution to the above linear SDE:

$$Y_t^i = \int_0^t -\sigma^i(s)X_0^{(i,0)}\Big(\sum_{j=i+1}^{N}\frac{\alpha X_0^{(j,0)}\sigma^j(s)}{1+\alpha X_0^{(j,0)}}\rho_{ij}\Big)ds + \int_0^t \sigma^i(s)X_0^{(i,0)}dW_s^i, \tag{3.4}$$

with $Y_0^i = 0$. □

Remark 6. *We parametrise the LIBOR market model in terms of the parameter ϵ_1 as follows:*

$$dL_t^{(i,\epsilon_1)} = \sigma^i(t)L_t^{(i,\epsilon_1)}\Big(-\sum_{j=i+1}^{N}\frac{\alpha X_t^{(j,\epsilon_1)}\sigma^j(t)}{1+\alpha X_t^{(j,\epsilon_1)}}\rho_{ij}dt + dW_t^i\Big).$$

and assume at $t=0$ that $L_0^{(i,\epsilon_1)} = X_0^{(i,\epsilon_1)}$ for all ϵ_1 and all $i=1,...,N$. If $\epsilon_1 = 1$, what we obtain is the standard LIBOR market model formulation and in particular $L_t^{(i,1)} = X_t^{(i,1)}$. For $\epsilon_1 = 0$, $X_t^{(i,0)}$ equals its starting value and thus the drift term in the following SDE is no longer stochastic:

$$dL_t^{(i,0)} = \sigma^i(t)L_t^{(i,0)}\Big(-\sum_{j=i+1}^{N}\frac{\alpha X_t^{(j,0)}\sigma^j(t)}{1+\alpha X_t^{(j,0)}}\rho_{ij}dt + dW_t^i\Big). \tag{3.5}$$

The next proposition provides a way for a pathwise approximation of $L_t^{(i,\epsilon_1)}$, by means of adjusting its SDE. This is achieved by adding $\mathbf{T}^n_{\epsilon_1}(X_t^{(j,\epsilon_1)})$ in the frozen drift part.

Proposition 1. *Assume the setup of Lemma 1 and assume further at $t=0$ that $L_0^{(i,\epsilon_1)} = X_0^{(i,\epsilon_1)}$ for all ϵ_1 and all $i=1,...,N$. Then the stochastic differential equation for $L_t^{(i,\epsilon_1)}$ with the unfrozen drift:*

$$dL_t^{(i,\epsilon_1)} = \sigma^i(t)L_t^{(i,\epsilon_1)}\Big(-\sum_{j=i+1}^{N}\frac{\alpha X_t^{(j,\epsilon_1)}\sigma^j(t)}{1+\alpha X_t^{(j,\epsilon_1)}}\rho_{ij}dt + dW_t^i\Big), \tag{3.6}$$

can be strongly approximated as $\epsilon_1 \downarrow 0$ by

$$d\hat{L}_t^{(i,\epsilon_1)} = \sigma^i(t)\hat{L}_t^{(i,\epsilon_1)}\Big(-\sum_{j=i+1}^{N}\frac{\alpha\big(\mathbf{T}^n_{\epsilon_1}(X_t^{(j,\epsilon_1)})\big)_+\sigma^j(t)}{1+\alpha\big(\mathbf{T}^n_{\epsilon_1}(X_t^{(j,\epsilon_1)})\big)_+}\rho_{ij}dt + dW_t^i\Big). \tag{3.7}$$

Remark 7. *For $n=0$, we derive the "freezing the drift" case. For $n=1$, we already obtain an improvement.*

Proof. First step is to interchange $X_t^{(j,\epsilon_1)}$ with $(X_t^{(j,\epsilon_1)})_+$ in (3.6) to obtain:

$$dL_t^{(i,\epsilon_1)} = \sigma^i(t)L_t^{(i,\epsilon_1)}\Big(-\sum_{j=i+1}^{N}\frac{\alpha(X_t^{(j,\epsilon_1)})_+\sigma^j(t)}{1+\alpha(X_t^{(j,\epsilon_1)})_+}\rho_{ij}dt + dW_t^i\Big).$$

This yields no change for the dynamics of $L_t^{(i,\epsilon_1)}$, since $X_t^{(j,\epsilon_1)} = (X_t^{(j,\epsilon_1)})_+$.

By Taylor's expansion, we know that as $\epsilon_1 \downarrow 0$, $\hat{L}_t^{(i,\epsilon_1)} \to L_t^{(i,\epsilon_1)}$ $\mathbb{P}$-a.s. The estimate for the error term is given by

$$\log \hat{L}_t^{(i,\epsilon_1)} - \log L_t^{(i,\epsilon_1)} =$$
$$= \int_0^t \sigma^i(s)\Big(- \sum_{j=i+1}^{N} \frac{\alpha\big(\mathbf{T}_{\epsilon_1}^n(X_s^{(j,\epsilon_1)})\big)_+ \sigma^j(s)}{1+\alpha\big(\mathbf{T}_{\epsilon_1}^n(X_s^{(j,\epsilon_1)})\big)_+}\rho_{ij} + \sum_{j=i+1}^{N} \frac{\alpha(X_t^{(j,\epsilon_1)})_+ \sigma^j(t)}{1+\alpha(X_t^{(j,\epsilon_1)})_+}\rho_{ij}\Big) ds \leq$$
$$\leq \int_0^t \alpha |X_s^{(j,\epsilon_1)} - (\mathbf{T}_{\epsilon_1}^n(X_s^{(j,\epsilon_1)}))_+| ds.$$

□

Remark 8. *The SDE for the approximated $\hat{L}_t^{(i,\epsilon_1)}$ is easier and faster to simulate than (3.1), as it is exhibited by the following example. Notice additionally that $\hat{L}_t^{(i,\epsilon_1)}$ is a continuous functional of the process Y_t^j* (3.4) *and of the Brownian path W_t^i. Eventually, by using $\hat{L}_t^{(i,\epsilon_1)}$ as the LIBOR rates, the computational complexity of the drift and thus of the model can be reduced substantially, while maintaining accuracy of prices.*

Example 2. *In this example, we examine the performance of the strong Taylor correction method. Let $N = 3$ and consider pricing a caplet on the LIBOR rate L^1 with strike K. Its price is given by:*

$$P_0^{cpt} = \alpha \mathbb{E}_{\mathbb{P}}\Big(\big(L_{T_1}^1 - K\big)_+\Big).$$

Assume that the volatility functions $\sigma^i(t) : [0, T_i] \to \mathbb{R}$ for $i = 1, 2, 3$ are given by (cf. Brigo and Mercurio [BM01]*, formulation (6.12)):*

$$\sigma^i(t) = \big(a(T_i - t) + d\big) \exp\big(-b(T_i - t)\big) + e,$$

where the constants a, b, d, e are the same for all three LIBOR rates and are equal to $a = -0.113035$, $b = 0.22911$, $d = -a$, $e = 0.684784$. Thus, we can write the model under the terminal measure $\mathbb{P}$ as:

$$dL_t^{(1,\epsilon_1)} = \sigma^1(t)L_t^{(1,\epsilon_1)}\Big(- \frac{\alpha X_t^{(2,\epsilon_1)}\sigma^2(t)\rho_{12}}{1+\alpha X_t^{(2,\epsilon_1)}} - \frac{\alpha X_t^{(3,\epsilon_1)}\sigma^3(t)\rho_{13}}{1+\alpha X_t^{(3,\epsilon_1)}}\Big)dt + \sigma^1(t)L_t^{(1,\epsilon_1)}dW_t^1,$$
$$dL_t^{(2,\epsilon_1)} = \sigma^2(t)L_t^{(2,\epsilon_1)}\Big(- \frac{\alpha X_t^{(3,\epsilon_1)}\sigma^3(t)\rho_{23}}{1+\alpha X_t^{(3,\epsilon_1)}}\Big)dt + \sigma^2(t)L_t^{(2,\epsilon_1)}dW_t^2,$$
$$dL_t^3 = \sigma^3(t)L_t^3 dW_t^3,$$
$$dX_t^{(1,\epsilon_1)} = \epsilon_1\Big(\sigma^1(t)X_t^{(1,\epsilon_1)}\Big(- \frac{\alpha X_t^{(2,\epsilon_1)}\sigma^2(t)\rho_{12}}{1+\alpha X_t^{(2,\epsilon_1)}} - \frac{\alpha X_t^{(3,\epsilon_1)}\sigma^3(t)\rho_{13}}{1+\alpha X_t^{(3,\epsilon_1)}}\Big)dt + \sigma^1(t)X_t^{(1,\epsilon_1)}dW_t^1\Big),$$
$$dX_t^{(2,\epsilon_1)} = \epsilon_1\Big(\sigma^2(t)X_t^{(2,\epsilon_1)}\Big(- \frac{\alpha X_t^{(3,\epsilon_1)}\sigma^3(t)\rho_{23}}{1+\alpha X_t^{(3,\epsilon_1)}}\Big)dt + \sigma^2(t)X_t^{(2,\epsilon_1)}dW_t^2\Big),$$
$$dX_t^{(3,\epsilon_1)} = \epsilon_1\Big(\sigma^3(t)X_t^{(3,\epsilon_1)}dW_t^3\Big),$$

with initial values $L_0^{(i,\epsilon_1)} = X_0^{(i,\epsilon_1)} = c_i$, for $i = 1, 2, 3$ and for all ϵ_1. The Brownian motion vector (W_t^1, W_t^2, W_t^3) is correlated with correlation coefficient ρ_{ij} given by:

$$\rho_{ij} = 0.49 + (1 - 0.49)\exp(-0.13|i - j|), \ i, j = 1, 2, 3.$$

The SDEs for the approximated LIBOR rates $\hat{L}_t^{(1,\epsilon_1)}$ *and* $\hat{L}_t^{(2,\epsilon_1)}$ *are given by:*

$$
\begin{aligned}
d\hat{L}_t^{(1,\epsilon_1)} &= \sigma^1(t)\hat{L}_t^{(1,\epsilon_1)}\Big(-\frac{\alpha\big(c_2+\epsilon_1 Y_t^2\big)_+\sigma^2(t)\rho_{12}}{1+\alpha\big(c_2+\epsilon_1 Y_t^2\big)_+} - \frac{\alpha\big(c_3+\epsilon_1 Y_t^3\big)_+\sigma^3(t)\rho_{13}}{1+\alpha\big(c_3+\epsilon_1 Y_t^3\big)_+}\Big)dt+ \\
&\quad + \sigma^1(t)\hat{L}_t^{(1,\epsilon_1)}dW_t^1, \\
d\hat{L}_t^{(2,\epsilon_1)} &= \sigma^2(t)\hat{L}_t^{(2,\epsilon_1)}\Big(-\frac{\alpha\big(c_3+\epsilon_1 Y_t^3\big)_+\sigma^3(t)\rho_{23}}{1+\alpha\big(c_3+\epsilon_1 Y_t^3\big)_+}\Big)dt + \sigma^2(t)\hat{L}_t^{(2,\epsilon_1)}dW_t^2.
\end{aligned}
$$

The partial derivative terms Y_t^2 *and* Y_t^3 *are equal to:*

$$
\begin{aligned}
Y_t^2 &= c_2\Big(\int_0^t \sigma^2(s)dW_s^2 - \frac{\alpha c_3\rho_{23}}{1+\alpha c_3}\int_0^t \sigma^2(s)\sigma^3(s)ds\Big), \\
Y_t^3 &= c_3\int_0^t \sigma^3(s)dW_s^3.
\end{aligned}
$$

We compare three caplet prices:

- *benchmark price, underlying* $L_t^{(1,\epsilon_1)}$*;*
- *strong Taylor price, underlying* $\hat{L}_t^{(1,\epsilon_1)}$*;*
- *frozen drift price, underlying* $L_t^{(1,0)}$*.*

Numerical results in basis points (bps) are displayed in Table 2 for parameters $\epsilon_1 = 1$, $N = 3$, $\alpha = 0.50137$, $c_1 = 3.86777\%$, $c_2 = 3.7574\%$, $c_3 = 3.8631\%$, $T_1 = 1.53151$, $T_i = T_1 + i\alpha$, $i = 2,3,4$. *We characteristically observe the difference in prices between the benchmark and frozen drift price, whilst our strong Taylor correction method performs very well and is computationally simpler and faster.*

strikes	*K=3%*	*K=3.5%*	*K=4%*	*K=5.75%*	*K=6.25%*	*K=8%*
benchmark	*11.1831*	*8.5897*	*6.5503*	*3.0349*	*2.4423*	*1.2969*
strong Taylor	*11.0687*	*8.5691*	*6.5867*	*3.1448*	*2.5513*	*1.3926*
frozen drift	*13.9551*	*11.1822*	*8.8803*	*4.6313*	*3.8506*	*2.2524*

Table 1: Caplet values in bps for parameters $\epsilon_1 = 1$, $\alpha = 0.50137$, $c_1 = 3.86777\%$, $c_2 = 3.7574\%$, $c_3 = 3.8631\%$ and $T_1 = 1.53151$.

3.3.2. *Weak Taylor Approximation.* In what follows, we provide some results on how to correct option prices obtained by the SDE with the frozen drift (3.5) by adding a correction term involving the appropriate Malliavin weight. Let $\mathbf{L}_{T_i}^{i,k,\epsilon_1}$ denote the vector of the LIBOR rates $(L_{T_i}^{(i,\epsilon_1)}, \ldots, L_{T_i}^{(k,\epsilon_1)})$.

Proposition 2. *Assume the setup of Lemma 1, where the* i^{th} *LIBOR rate is given by:*

$$
dL_t^{(i,\epsilon_1)} = \sigma^i(t)L_t^{(i,\epsilon_1)}\Big(-\sum_{j=i+1}^{N}\frac{\alpha X_t^{(j,\epsilon_1)}\sigma^j(t)}{1+\alpha X_t^{(j,\epsilon_1)}}\rho_{ij}dt + dW_t^i\Big). \tag{3.8}
$$

with $L_0^{(i,\epsilon_1)} = X_0^{(i,\epsilon_1)}$ *for all* ϵ_1 *and all* $i = 1,...,N$*. Assume furthermore that the Malliavin covariance matrix* $\gamma(\mathbf{L}_{T_i}^{i,k,0})$ *is invertible. Then the price of an option with*

payoff $g(\mathbf{L}_{T_i}^{i,k,\epsilon_1})$, *for* $i \leq k \leq N$ *and* g *bounded measurable, can be approximated by the weak Taylor approximation of order one:*

$$\mathbf{W}_a^1(g, \mathbf{L}_{T_i}^{i,k,\epsilon_1}) = P(0,T)\Big(\mathbb{E}_{\mathbb{P}}\big(g(\mathbf{L}_{T_i}^{i,k,0})\big) + \epsilon_1 \mathbb{E}_{\mathbb{P}}\big(g(\mathbf{L}_{T_i}^{i,k,0})\zeta_{T_i}\big)\Big), \tag{3.9}$$

where the Malliavin weight ζ_{T_i} *is given by:*

$$\zeta_{T_i} = \delta\Big((D_t \mathbf{L}_{T_i}^{i,k,0})^{\mathrm{T}} \gamma^{-1}(\mathbf{L}_{T_i}^{i,k,0}) \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} \mathbf{L}_{T_i}^{i,k,\epsilon_1}\Big), \tag{3.10}$$

for $t \leq T_i$.

Proof. The weight ζ_{T_i} is obtained by (2.4). Notice that we can write:

$$\frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} \mathbb{E}_{\mathbb{P}}\big(g(\mathbf{L}_{T_i}^{i,k,\epsilon_1})\big) = \mathbb{E}_{\mathbb{P}}\big(g\big(\mathbf{L}_{T_i}^{i,k,0}\big)\zeta_{T_i}\big),$$

and hence the result (3.9) by Definition 2 for $n = 1$. □

Example 3. *In this example we let* $N = 3$ *and we price a payers swaption with strike price* K *and maturity* T_1, *where the underlying swap is entered at* T_1 *and has payment dates* T_2 *and* T_3. *We assume that the volatility functions* $\sigma^i(t) : [0, T_i] \to \mathbb{R}$ *for* $i = 1, 2, 3$ *are constant:*

$$\sigma^1(t) = \sigma_1, \sigma^2(t) = \sigma_2, \sigma^3(t) = \sigma_3,$$

such that we obtain under the terminal measure $\mathbb{P}$:

$$\begin{aligned}
dL_t^{(1,\epsilon_1)} &= \sigma_1 L_t^{(1,\epsilon_1)}\Big(\rho_{12}\Big(-\frac{\alpha X_t^{(2,\epsilon_1)}\sigma_2}{1+\alpha X_t^{(2,\epsilon_1)}} - \frac{\alpha X_t^{(3,\epsilon_1)}\sigma_3}{1+\alpha X_t^{(3,\epsilon_1)}}\Big)dt + dW_t^1\Big),\\
dL_t^{(2,\epsilon_1)} &= \sigma_2 L_t^{(2,\epsilon_1)}\Big(-\frac{\alpha X_t^{(3,\epsilon_1)}\sigma_3}{1+\alpha X_t^{(3,\epsilon_1)}}dt + dW_t^2\Big),\\
dL_t^3 &= \sigma_3 L_t^3 dW_t^2, \qquad (3.11)\\
dX_t^{(1,\epsilon_1)} &= \epsilon_1\Big(\sigma_1 X_t^{(1,\epsilon_1)}\Big(\rho_{12}\Big(-\frac{\alpha X_t^{(2,\epsilon_1)}\sigma_2}{1+\alpha X_t^{(2,\epsilon_1)}} - \frac{\alpha X_t^{(3,\epsilon_1)}\sigma_3}{1+\alpha X_t^{(3,\epsilon_1)}}\Big)dt + dW_t^1\Big)\Big),\\
dX_t^{(2,\epsilon_1)} &= \epsilon_1\Big(\sigma_2 X_t^{(2,\epsilon_1)}\Big(-\frac{\alpha X_t^{(3,\epsilon_1)}\sigma_3}{1+\alpha X_t^{(3,\epsilon_1)}}dt + dW_t^2\Big)\Big),\\
dX_t^{(3,\epsilon_1)} &= \epsilon_1\Big(\sigma_3 X_t^{(3,\epsilon_1)} dW_t^2\Big),
\end{aligned}$$

with initial values $L_0^{(i,\epsilon_1)} = X_0^{(i,\epsilon_1)} = c_i$, *for* $i = 1, 2, 3$ *and for all* ϵ_1. W_t^1 *and* W_t^2 *are correlated with correlation coefficient* ρ_{12}. *We freeze the drifts in the above equations to obtain:*

$$\begin{aligned}
L_t^{(1,0)} &= c_1 \exp\Big(\sigma_1 W_t^1 - \Big(\rho_{12}\big(\frac{\alpha c_2 \sigma_2}{1+\alpha c_2} + \frac{\alpha c_3\sigma_3}{1+\alpha c_3}\big) + \frac{1}{2}\sigma_1\Big)\sigma_1 t\Big),\\
L_t^{(2,0)} &= c_2 \exp\Big(\sigma_2 W_t^2 - \Big(\frac{\alpha c_3\sigma_3}{1+\alpha c_3} + \frac{1}{2}\sigma_2\Big)\sigma_2 t\Big),\\
L_t^3 &= c_3 \exp\Big(\sigma_3 W_t^2 - \frac{1}{2}\sigma_3^2 t\Big).
\end{aligned}$$

Similarly to the previous example, we compare four option prices:

- *benchmark price;*
- *frozen drift;*

- *strong Taylor price;*
- *weak Taylor price.*

The weak correction formula (3.9) *adds a correction term to the closed form price of the option. The swaption payoff at* T_i *can be found for example in* [MR98]*:*

$$P_{T_i}^{swptn} = \Big(1 - \sum_{k=i+1}^{N+1} \alpha_k \prod_{j=i}^{k-1} (1 + \alpha L_{T_i}^j)^{-1}\Big)_+,$$

if the underlying swap is entered at time T_i *and has payment dates* $T_{i+1}, ..., T$. α_k *is given by:*

$$\alpha_k = \begin{cases} K\alpha, & k = i+1, \dots, N, \\ 1 + K\alpha, & k = N+1. \end{cases}$$

The payers swaption value at time $t = 0$ *can be written as:*

$$\begin{aligned} P_0^{swptn} &= P(0,T_i)\mathbb{E}_{\mathbb{P}^i}\big(P_{T_i}^{swptn}\big) = \\ &= P(0,T)\mathbb{E}_{\mathbb{P}}\Big(\Big(-\sum_{k=i}^{N} \alpha_k \prod_{j=k}^{N} (1 + \alpha L_{T_i}^j) - (1 + K\alpha)\Big)_+\Big), \end{aligned} \tag{3.12}$$

where $\alpha_i := -1$ *and* $\mathbb{P}^i$ *denotes the forward measure corresponding to the bond* $P(t, T_i)$ *as numéraire. Therefore, its benchmark price is given by the above formula with* $N = 2$ *and* $i = 1$*:*

$$bP_0^{swptn} = P(0,T)\Bigg(\mathbb{E}_{\mathbb{P}}\Big(\Big(\alpha L_{T_1}^1 + \alpha L_{T_1}^2 + \alpha^2 L_{T_1}^1 L_{T_1}^2 - K\alpha^2 L_{T_1}^2 - 2K\alpha\Big)_+\Big)\Bigg).$$

Its weak Taylor price is given by (3.9) *with* $i = 1$ *and* $k = N = 2$.

$$\begin{aligned} wP_0^{swptn} = P(0,T)\Bigg(&\mathbb{E}_{\mathbb{P}}\Big(\Big(\alpha L_{T_1}^{(1,0)} + \alpha L_{T_1}^{(2,0)} + \alpha^2 L_{T_1}^{(1,0)} L_{T_1}^{(2,0)} - K\alpha^2 L_{T_1}^{(2,0)} - 2K\alpha\Big)_+\Big) + \\ &+ \epsilon_1 \mathbb{E}_{\mathbb{P}}\Big(\Big(\alpha L_{T_1}^{(1,0)} + \alpha L_{T_1}^{(2,0)} + \alpha^2 L_{T_1}^{(1,0)} L_{T_1}^{(2,0)} - K\alpha^2 L_{T_1}^{(2,0)} - 2K\alpha\Big)_+ \zeta_{T_1}\Big)\Bigg). \end{aligned}$$

The weight ζ_{T_1} *is given by* (3.10). *The partial derivative terms* $C_{T_1}^1 := \frac{\partial}{\partial \epsilon_1}|_{\epsilon_1 = 0} L_{T_1}^{(1,\epsilon_1)}$ *and* $C_{T_1}^2 := \frac{\partial}{\partial \epsilon_1}|_{\epsilon_1=0} L_{T_1}^{(2,\epsilon_1)}$ *are given by:*

$$C_{T_1}^1 = L_{T_1}^{(1,0)} \int_0^{T_1} \sigma_1 \rho_{12} \Big(\frac{\sigma_3 \alpha c_3 \beta_2}{(1 + \alpha c_3)} t - (\beta_2 + \beta_3) W_t^2\Big) dt,$$

and:

$$C_{T_1}^2 = L_{T_1}^{(2,0)} \int_0^{T_1} -\sigma_2 \beta_3 W_t^2 dt,$$

with $C_0^1 = C_0^2 = 0$ *and* $\beta_2 := \frac{\alpha c_2^2 \sigma_2^2}{(1+\alpha c_2)^2}$, $\beta_3 := \frac{\alpha c_3^2 \sigma_3^2}{(1+\alpha c_3)^2}$. *The Malliavin covariance matrix of the vector* $(L_{T_1}^{(1,0)}, L_{T_1}^{(2,0)})$ *is equal to:*

$$\begin{aligned} &\gamma\big((L_{T_1}^{(1,0)}, L_{T_1}^{(2,0)})\big) = \begin{pmatrix} (1 + \rho_{12}^2)(L_{T_1}^{(1,0)})^2 T_1 \sigma_1^2 & 2\rho_{12}(L_{T_1}^{(1,0)})(L_{T_1}^{(2,0)}) T_1 \sigma_1 \sigma_2 \\ 2\rho_{12}(L_{T_1}^{(1,0)})(L_{T_1}^{(2,0)}) T_1 \sigma_1 \sigma_2 & (1 + \rho_{12}^2)(L_{T_1}^{(2,0)})^2 T_1 \sigma_2^2 \end{pmatrix} \\ &\Rightarrow det\Big(\gamma\big((L_{T_1}^{(1,0)}, L_{T_1}^{(2,0)})\big)\Big) = (L_{T_1}^{(1,0)})^2 (L_{T_1}^{(2,0)})^2 T_1^2 \sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2). \end{aligned}$$

The determinant is not zero as long as $\rho_{12} \neq 1$, which is a natural assumption. Hence under this condition, its inverse is given by:

$$\gamma^{-1}\big((L^{(1,0)}_{T_1}, L^{(2,0)}_{T_1})\big) = \frac{1}{(1-\rho_{12}^2)} \begin{pmatrix} \frac{1+\rho_{12}^2}{(L^{(1,0)}_{T_1})^2 T_1 \sigma_1^2} & -\frac{2\rho_{12}}{L^{(1,0)}_{T_1} L^{(2,0)}_{T_1} T_1 \sigma_1 \sigma_2} \\ -\frac{2\rho_{12}}{L^{(1,0)}_{T_1} L^{(2,0)}_{T_1} T_1 \sigma_1 \sigma_2} & \frac{1+\rho_{12}^2}{(L^{(2,0)}_{T_1})^2 T_1 \sigma_2^2} \end{pmatrix}.$$

Write the weight $\zeta_{T_1} = \zeta^1_{T_1} + \zeta^2_{T_1}$, where the first weight $\zeta^1_{T_1}$ is obtained as:

$$\zeta^1_{T_1} = \int_0^{T_1} \Big(D^1_t L^{(1,0)}_{T_1}\big(C^1_{T_1}\gamma^{-1}_{11} + C^2_{T_1}\gamma^{-1}_{12}\big) + D^1_t L^{(2,0)}_{T_1}\big(C^1_{T_1}\gamma^{-1}_{21} + C^2_{T_1}\gamma^{-1}_{22}\big)\Big)\delta W^1_t,$$

and $\zeta^2_{T_1}$ similarly:

$$\zeta^2_{T_1} = \int_0^{T_1} \Big(D^2_t L^{(1,0)}_{T_1}\big(C^1_{T_1}\gamma^{-1}_{11} + C^2_{T_1}\gamma^{-1}_{12}\big) + D^2_t L^{(2,0)}_{T_1}\big(C^1_{T_1}\gamma^{-1}_{21} + C^2_{T_1}\gamma^{-1}_{22}\big)\Big)\delta W^2_t.$$

Performing all necessary calculations, we conclude that:

$$\zeta^1_{T_1} = \rho_{12}\Big(W^1_{T_1}\Big(\frac{\sigma_3\alpha c_3\beta_2 T_1}{2(1+\alpha c_3)} - \frac{(\beta_2+\beta_3)}{T_1}\int_0^{T_1} W^2_t dt\Big) + \frac{\rho_{12}(\beta_2+\beta_3)T_1}{2}\Big) - \\ - \rho_{12}\Big(\frac{\rho_{12}\beta_3 T_1}{2} - \frac{\beta_3 W^1_{T_1}}{T_1}\int_0^{T_1} W^2_t dt\Big).$$

Analogously we obtain $\zeta^2_{T_1}$ as:

$$\zeta^2_{T_1} = \rho_{12}^2\Bigg(W^2_{T_1}\Big(\frac{\sigma_3\alpha c_3\beta_2 T_1}{2(1+\alpha c_3)} - \frac{(\beta_2+\beta_3)}{T_1}\int_0^{T_1} W^2_t dt\Big) + \frac{(\beta_2+\beta_3)T_1}{2}\Bigg) - \\ - \Big(\frac{\beta_3 T_1}{2} - \frac{\beta_3 W^2_{T_1}}{T_1}\int_0^{T_1} W^2_t dt\Big).$$

Notice that the weights are functions of normal variables and thus the calculation of the weak Taylor price amounts just to computation of deterministic integrals. Table 3 gives the swaption prices in bps for parameters $N = 3$, $\alpha = 0.25$, $\sigma_1 = 18\%$, $\sigma_2 = 15\%$, $\sigma_3 = 12\%$, $c_0 = 5.28875\%$, $c_1 = 5.37375\%$, $c_2 = 5.40\%$, $c_3 = 5.40125\%$ and $\rho_{12} = 0.75$.

strikes	*K=4%*	*K=4.5%*	*K=4.75%*	*K=5%*	*K=5.15%*	*K=5.25%*
benchmark	*10.2240*	*6.5386*	*4.7454*	*3.1060*	*2.2599*	*1.7758*
frozen drift	*10.2132*	*6.5326*	*4.7419*	*3.1028*	*2.2582*	*1.7618*
strong Taylor	*10.2240*	*6.5386*	*4.7454*	*3.1060*	*2.2599*	*1.7758*
weak Taylor	*10.2266*	*6.5407*	*4.7485*	*3.1064*	*2.2593*	*1.7626*

Table 2: Swaption values in bps for parameters $\epsilon_1 = 1$, $\alpha = 0.25$, $\sigma_1 = 18\%$, $\sigma_2 = 15\%$, $\sigma_3 = 12\%$, $c_0 = 5.28875\%$, $c_1 = 5.37375\%$, $c_2 = 5.40\%$, $c_3 = 5.40125\%$ and $\rho_{12} = 0.75$.

3.4. **The Stochastic Volatility LIBOR Market Model.** In this section, we develop a stochastic volatility LMM. The stochastic volatility parameter v_t follows a square root process, like in the extensively applied Heston model [Hes93]. The resulting model, called hereafter the *stochastic volatility LMM* (SVLMM), has the following dynamics under the terminal measure:

$$\begin{aligned} dL_t^i &= \sigma^i(t) L_t^i \sqrt{v_t}\Big(- \sum_{j=i+1}^{N} \frac{\alpha L_t^j \sigma^j(t)}{1+\alpha L_t^j}\rho_{ij}\sqrt{v_t}dt + dW_t^i\Big), i = 1, ..., N, \\ dv_t &= \kappa(\theta - v_t)dt + \epsilon_2 \sqrt{v_t} dB_t, \end{aligned} \tag{3.13}$$

where $\kappa, \theta, \epsilon_2 \in \mathbb{R}_+$. The Brownian motions $W_t = (W_t^1, ..., W_t^N)$ and B_t are expressed under the terminal measure with correlations $dW_t^i dB_t = \rho_i dt$ and $dW_t^i dW_t^j = \rho_{ij} dt$ for $i, j = 1, ...N$. We assume additionally that the filtration $(\mathcal{F}_t)_{0 \le t \le T}$ is generated by both Brownian motions. Observe that the process v_t is a time-changed squared Bessel process with dimension $\delta = 4\kappa\theta/\epsilon_2^2$. If $\delta \ge 2$, then the point zero is unattainable. So we require $2\kappa\theta \ge \epsilon_2^2$ for the process v_t not to reach zero.

3.4.1. *Pricing a multi-LIBOR option.* In this section, we aim at approximating the price of an option with payoff depending on the vector $\mathbf{L}_{T_i}^{i,k,\epsilon_1,\epsilon_2} = (L_{T_i}^{i,\epsilon_1,\epsilon_2}, \dots, L_{T_i}^{k,\epsilon_1,\epsilon_2})$. We interpret the volatility of the volatility parameter ϵ_2 as a parameter on which the LIBOR rates depend. Overall, we parametrise the SVLMM by both ϵ_1 and ϵ_2 and correct prices in a weak sense introducing Malliavin weights.

Proposition 3. *Consider the SVLMM* (3.13) *and assume that the Malliavin covariance matrix* $\gamma(\mathbf{L}_{T_i}^{i,k,0,0})$ *is invertible. Then the price of an option with payoff* $\psi(\mathbf{L}_{T_i}^{i,k,\epsilon_1,\epsilon_2})$, $i \le k \le N$, *where* ψ *is a bounded measurable function, can be approximated by the weak Taylor approximation of order one:*

$$\begin{aligned} \mathbf{W}^1_{(\epsilon_1,\epsilon_2)}(\psi, \mathbf{L}_{T_i}^{i,k,\epsilon_1,\epsilon_2})) = P(0,T)\Big(&\mathbb{E}_{\mathbb{P}}\big(\psi(\mathbf{L}_{T_i}^{i,k,0,0})\big) + \epsilon_1 \mathbb{E}_{\mathbb{P}}\big(\psi(\mathbf{L}_{T_i}^{i,k,0,0})\zeta_{T_i}\big) + \\ &+ \epsilon_2 \mathbb{E}_{\mathbb{P}}\big(\psi(\mathbf{L}_{T_i}^{i,k,0,0})\pi_{T_i}\big)\Big), \end{aligned} \tag{3.14}$$

where the Malliavin weights ζ_{T_i}, π_{T_i} *are given by:*

$$\zeta_{T_i} = \delta\Big((D_t \mathbf{L}_{T_i}^{i,k,0,0})^{\mathrm{T}} \gamma^{-1}(\mathbf{L}_{T_i}^{i,k,0,0}) \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} \mathbf{L}_{T_i}^{i,k,\epsilon_1,0}\Big), \tag{3.15}$$

$$\pi_{T_i} = \delta\Big((D_t \mathbf{L}_{T_i}^{i,k,0,0})^{\mathrm{T}} \gamma^{-1}(\mathbf{L}_{T_i}^{i,k,0,0}) \frac{\partial}{\partial \epsilon_2}\Big|_{\epsilon_2=0} \mathbf{L}_{T_i}^{i,k,0,\epsilon_2}), \tag{3.16}$$

for $t \le T_i$.

Proof. The weights ζ_{T_i} and π_{T_i} are obtained by (2.4). We derive (3.14) by noticing that:

$$\begin{aligned} &\mathbb{E}\big(\psi(\mathbf{L}_{T_i}^{i,k,\epsilon_1,\epsilon_2})\big) \simeq \\ &= \mathbb{E}\big(\psi(\mathbf{L}_{T_i}^{i,k,0,0})\big) + \epsilon_1 \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} \mathbb{E}\big(\psi(\mathbf{L}_{T_i}^{i,k,\epsilon_1,0})\big) + \epsilon_2 \frac{\partial}{\partial \epsilon_2}\Big|_{\epsilon_2=0} \mathbb{E}\big(\psi(\mathbf{L}_{T_i}^{i,k,0,\epsilon_2})\big) = \\ &= \mathbb{E}\big(\psi(\mathbf{L}_{T_i}^{i,k,0,0})\big) + \epsilon_1 \mathbb{E}\big(\psi(\mathbf{L}_{T_i}^{i,k,0,0})\zeta_{T_i}\big) + \epsilon_2 \mathbb{E}\big(\psi(\mathbf{L}_{T_i}^{i,k,0,0})\pi_{T_i}\big), \end{aligned}$$

from Definition 2 for $n = 1$. □

Example 4. *Let $N = 2$ and consider the SVLMM where the volatility functions $\sigma^i(t) : [0, T_i] \to \mathbb{R}$ for $i = 1, 2$ are assumed to be constant and in particular $\sigma^1(t) = \sigma_1$, $\sigma^2(t) = \sigma_2$. We derive an approximative formula for the price of a payers swaption with maturity T_1 and strike price K. The underlying swap is entered at T_1 and has payment dates T_2, T_3. Under the terminal measure $\mathbb{P}$ we can write the SDEs for the LIBOR rates and stochastic volatility as:*

$$\begin{aligned}
dv_t^{\epsilon_2} &= \kappa\Big(\theta - v_t^{\epsilon_2}\Big)dt + \epsilon_2\sqrt{v_t^{\epsilon_2}}dB_t,\\
dL_t^{(1,\epsilon_1,\epsilon_2)} &= -L_t^{(1,\epsilon_1,\epsilon_2)}\rho_{12}\frac{\alpha X_t^{(2,\epsilon_1,\epsilon_2)}\sigma_2}{1+\alpha X_t^{(2,\epsilon_1,\epsilon_2)}}\sigma_1 v_t^{\epsilon_2}dt + \sigma_1 L_t^{(1,\epsilon_1,\epsilon_2)}\sqrt{v_t^{\epsilon_2}}dW_t^1,\\
dL_t^{(2,\epsilon_2)} &= \sigma_2 L_t^{(2,\epsilon_2)}\sqrt{v_t^{\epsilon_2}}dW_t^2,\\
dX_t^{(2,\epsilon_1)} &= \epsilon_1\Big(\sigma_2 X_t^{(2,\epsilon_2)}\sqrt{v_t^{\epsilon_2}}dW_t^2\Big).
\end{aligned}$$

W_t^1 and W_t^2 are assumed to be correlated, so correlations are as $dW_t^i dB_t = \rho_i dt$ and $dW_t^1 dW_t^2 = \rho_{12}$ for $i = 1, 2$. The $(0,0)$-model is given by:

$$\begin{aligned}
v_t^0 &= \exp(-\kappa t)(v_0^0 - \theta) + \theta,\\
L_{T_1}^{(1,0,0)} &= c_1\exp\Big(\sigma_1\int_0^{T_1}\sqrt{v_t^0}dW_t^1 - \big(\frac{\alpha c_2\rho_{12}}{1+\alpha c_2}\sigma_2 + \frac{1}{2}\sigma_1\big)c\sigma_1\Big),\\
L_{T_1}^{(2,0)} &= c_2\exp\Big(\sigma_2\int_0^{T_1}\sqrt{v_t^0}dW_t^2 - \frac{1}{2}\sigma_2^2 c\Big),\\
X_t^{(2,0,0)} &= c_2,
\end{aligned}$$

with $c := \int_0^{T_1} v_t^0 dt = \theta T_1 - \frac{v_0^0-\theta}{\kappa}(\exp(-\kappa T_1) - 1)$. As in the previous example, we compare the following option prices:

- *benchmark price;*
- *frozen drift;*
- *weak Taylor price* (3.14).

The benchmark price is given by (3.12) *with $N = 2$ and $i = 1$:*

$$bP_0^{swptn} = P(0,T)\mathbb{E}_{\mathbb{P}}\Big(\big(\alpha L_{T_1}^{(1,\epsilon_1,\epsilon_2)} + \alpha L_{T_1}^{(2,\epsilon_2)} + \alpha^2 L_{T_1}^{(1,\epsilon_1,\epsilon_2)}L_{T_1}^{(2,\epsilon_2)} - K\alpha^2 L_{T_1}^{(2,\epsilon_2)} - 2K\alpha\big)_+\Big).$$

The weak Taylor price is obtained by (3.14):

$$\begin{aligned}
wP_0^{swptn} = P(0,T)\Bigg(&\mathbb{E}_{\mathbb{P}}\Big(\big(\alpha L_{T_1}^{(1,0,0)} + \alpha L_{T_1}^{(2,0)} + \alpha^2 L_{T_1}^{(1,0,0)}L_{T_1}^{(2,0)} - K\alpha^2 L_{T_1}^{(2,0)} -\\
&- 2K\alpha\big)_+\Big) + \epsilon_1\mathbb{E}_{\mathbb{P}}\Big(\big(\alpha L_{T_1}^{(1,0,0)} + \alpha L_{T_1}^{(2,0)} + \alpha^2 L_{T_1}^{(1,0,0)}L_{T_1}^{(2,0)} - K\alpha^2\cdot\\
&\cdot L_{T_1}^{(2,0)} - 2K\alpha\big)_+\zeta_{T_1}\Big) + \epsilon_2\mathbb{E}_{\mathbb{P}}\Big(\big(\alpha L_{T_1}^{(1,0,0)} + \alpha L_{T_1}^{(2,0)} + \alpha^2 L_{T_1}^{(1,0,0)}L_{T_1}^{(2,0)} -\\
&- K\alpha^2 L_{T_1}^{(2,0)} - 2K\alpha\big)_+\pi_{T_1}\Big)\Bigg).
\end{aligned}$$

We calculate the Malliavin weights ζ_{T_1}, π_{T_1} as given by (3.15) *and* (3.16) *correspondingly. We can express the weight ζ_{T_1} as:*

$$\zeta_{T_1} = \zeta_{T_1}^1 + \zeta_{T_1}^2,$$

with:

$$\zeta^1_{T_1} = \int_0^{T_1} \Bigg(\Big(D^1_t L^{(1,0,0)}_{T_1} \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} L^{(1,\epsilon_1,0)}_{T_1} \gamma^{-1}(L^{(1,0,0)}_{T_1}, L^{(2,0,0)}_{T_1})_{11} \Big) + \\ + \Big(D^1_t L^{(2,0,0)}_{T_1} \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} L^{(1,\epsilon_1,0)}_{T_1} \gamma^{-1}(L^{(1,0,0)}_{T_1}, L^{(2,0,0)}_{T_1})_{21} \Big) \Bigg) \delta W^1_t,$$

and:

$$\zeta^2_{T_1} = \int_0^{T_1} \Bigg(\Big(D^2_t L^{(1,0,0)}_{T_1} \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} L^{(1,\epsilon_1,0)}_{T_1} \gamma^{-1}(L^{(1,0,0)}_{T_1}, L^{(2,0,0)}_{T_1})_{11} \Big) + \\ + \Big(D^2_t L^{(2,0,0)}_{T_1} \frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} L^{(1,\epsilon_1,0)}_{T_1} \gamma^{-1}(L^{(1,0,0)}_{T_1}, L^{(2,0,0)}_{T_1})_{21} \Big) \Bigg) \delta W^2_t.$$

since $\frac{\partial}{\partial \epsilon_1}|_{\epsilon_1=0}\ L^{(2,\epsilon_1,0)}_{T_1} = 0$. *The partial derivative term with respect to* ϵ_1 *for* L^1 *is given by:*

$$\frac{\partial}{\partial \epsilon_1}\Big|_{\epsilon_1=0} L^{(1,\epsilon_1,0)}_{T_1} = L^{(1,0,0)}_{T_1} \int_0^{T_1} \sigma_1 v^0_t \Big(- \frac{\alpha \sigma_2^2 c_2^2 \rho_{12}}{(1+\alpha c_2)^2} \int_0^t \sqrt{v^0_s} dW^2_s \Big) dt = \\ = -\sigma_1 \rho_{12} \beta_2 L^{(1,0,0)}_{T_1} \int_0^{T_1} \sqrt{v^0_t} \Big(\theta (T_1 - t) - \frac{v^0_0 - \theta}{\kappa} (\exp(-\kappa T_1) - \exp(-\kappa t)) \Big) dW^2_t,$$

where $\beta_2 = \frac{\alpha \sigma_2^2 c_2^2}{(1+\alpha c_2)^2}$. *Similarly the weight* π_{T_1} *is given by:*

$$\pi_{T_1} = \pi^1_{T_1} + \pi^2_{T_1},$$

with:

$$\pi^1_{T_1} = \int_0^{T_1} \Big(\sum_{l=1}^2 D^1_t L^{(l,0,0)}_{T_1} \sum_{j=1}^2 \frac{\partial}{\partial \epsilon_2}\Big|_{\epsilon_2=0} L^{(j,0,\epsilon_2)}_{T_1} \gamma^{-1} \big((L^{(1,0,0)}_{T_1}, L^{(2,0,0)}_{T_1})_{lj} \big) \Big) \delta W^1_t,$$

and:

$$\pi^2_{T_1} = \int_0^{T_1} \Big(\sum_{l=1}^2 D^2_t L^{(l,0,0)}_{T_1} \sum_{j=1}^2 \frac{\partial}{\partial \epsilon_2}\Big|_{\epsilon_2=0} L^{(j,0,\epsilon_2)}_{T_1} \gamma^{-1} \big((L^{(1,0,0)}_{T_1}, L^{(2,0,0)}_{T_1})_{lj} \big) \Big) \delta W^2_t.$$

Partial derivative terms are equal to:

$$\frac{\partial}{\partial \epsilon_2}\Big|_{\epsilon_2=0} L^{(1,0,\epsilon_2)}_{T_1} = L^{(1,0,0)}_{T_1} \Big(\frac{\sigma_1}{2} \int_0^{T_1} \frac{\exp(-\kappa t)}{\sqrt{v^0_t}} \int_0^t \exp(\kappa s) \sqrt{v^0_s} dB_s dW^1_t + \\ + \frac{1}{\kappa} \big(\frac{\sigma_1^2}{2} + \frac{\alpha c_2 \sigma_1 \sigma_2 \rho_{12}}{1 + \alpha c_2} \big) \int_0^{T_1} \exp(\kappa s) \sqrt{v^0_s} \big(\exp(-\kappa T_1) - \exp(-\kappa s) \big) dB_s \Big).$$

Doing similar calculations, we derive the second partial derivative:

$$\frac{\partial}{\partial \epsilon_2}\Big|_{\epsilon_2=0} L^{(2,\epsilon_2)}_{T_1} = L^{(2,0)}_{T_1} \Big(\frac{1}{2} \int_0^{T_1} \frac{\sigma_2}{\sqrt{v^0_t}} V_t dW^2_t - \frac{1}{2} \int_0^{T_1} \sigma_1 \sigma_2 V_t dt \Big),$$

where $V_t = \exp(-\kappa t) \int_0^t \exp(\kappa s) \sqrt{v^0_s} dB_s$.

We calculate the Malliavin covariance matrix $\gamma\big((L_{T_1}^{(1,0,0)}, L_{T_1}^{(2,0,0)})\big)$ and its inverse.

$$\gamma = \begin{pmatrix} (1+\rho_{12}^2)(L_{T_1}^{(1,0,0)})^2\sigma_1^2 \underbrace{\int_0^{T_1} v_t^0 dt}_{=c} & 2\rho_{12}L_{T_1}^{(1,0,0)}L_{T_1}^{(2,0,0)}\sigma_1\sigma_2 c \\ \\ 2\rho_{12}L_{T_1}^{(1,0,0)}L_{T_1}^{(2,0,0)}\sigma_1\sigma_2 c & (1+\rho_{12}^2)(L_{T_1}^{(2,0,0)})^2\sigma_2^2 c \end{pmatrix}$$

$$\Rightarrow det\Big(\gamma\big((L_{T_1}^{(1,0,0)}, L_{T_1}^{(2,0,0)})\big)\Big) = (L_{T_1}^{(1,0,0)})^2(L_{T_1}^{(2,0,0)})^2\sigma_1^2\sigma_2^2c^2(1-\rho_{12}^2).$$

Hence its inverse is given by, for $\rho_{12} \neq 1$:

$$\gamma^{-1} = \frac{1}{(1-\rho_{12}^2)}\begin{pmatrix} \frac{1+\rho_{12}^2}{(L_{T_1}^{(1,0,0)})^2\sigma_1^2 c} & -\frac{2\rho_{12}}{L_{T_1}^{(1,0,0)}L_{T_1}^{(2,0,0)}\sigma_1\sigma_2 c} \\ \\ -\frac{2\rho_{12}}{L_{T_1}^{(1,0,0)}L_{T_1}^{(2,0,0)}\sigma_1\sigma_2 c} & \frac{1+\rho_{12}^2}{(L_{T_1}^{(2,0,0)})^2\sigma_2^2 c} \end{pmatrix}.$$

If we define $X_i = \int_0^{T_1}\sqrt{v_t^0}dW_t^i$, $i = 1,2$ and $Y = \int_0^{T_1}\sqrt{v_t^0}\Big(\theta(T_1-t) - \frac{v_0^0-\theta}{\kappa}(\exp(-\kappa T_1) - \exp(-\kappa t))\Big)dW_t^2$, we finally obtain the weights as:

$$\zeta_{T_1}^1 = -\frac{\rho_{12}\beta_2}{c}\Big(X_1Y - \mathbb{C}ov(X_1, Y)\Big),$$

and:

$$\zeta_{T_1}^2 = \frac{\rho_{12}^2\beta_2}{c}\Big(X_2Y - \mathbb{C}ov(X_2, Y)\Big).$$

Moreover, for the weight π_{T_1} we define:

$$B = \int_0^{T_1} g(t)\Big(\exp(-\kappa T_1) - \exp(-\kappa t)\Big)dB_t,$$

and random variables D_i, Z_i for $i = 1,2$:

$$D_i = \int_0^{T_1} f(t)\int_0^t g(s)dW_s^i dW_t^i,$$
$$Z_i = \int_0^{T_1} f(t)\int_0^t g(s)dZ_s^i dW_t^i,$$

where the Brownian motions Z_t^i are independent from W_t^i and $f(t) = \frac{\exp(-\kappa t)}{\sqrt{v_t^0}}$, $g(s) = \exp(\kappa s)\sqrt{v_s^0}$. Therefore, we obtain the weights as:

$$\pi_{T_1}^1 = \frac{1}{2c}\Bigg(X_1(\rho_1 D_1 + \sqrt{1-\rho_1^2}Z_1) + \frac{BX_1}{\kappa}\Big(\frac{\alpha c_2(2\rho_{12}\sigma_2 + \sigma_1) + \sigma_1}{1+\alpha c_2}\Big) -$$
$$- \frac{\rho_1 E}{\kappa}\Big(\frac{\alpha c_2(2\rho_{12}\sigma_2 + \sigma_1) + \sigma_1}{1+\alpha c_2}\Big)\Bigg) - \frac{\rho_{12}}{2c}\Bigg(X_1(\rho_2 D_2 + \sqrt{1-\rho_2^2}Z_2) +$$
$$+ \frac{\sigma_1 BX_1}{\kappa} + \frac{\rho_{12}B}{\kappa} - \frac{\sigma_1\rho_1 E}{\kappa}\Bigg),$$

where E equals to $\Big(\frac{1}{\kappa}\big(1-\exp(-\kappa T_1)\big)-T_1\Big)$. *Similarly we get* $\pi^2_{T_1}$ *as:*

$$\pi^2_{T_1} = \frac{1}{2c}\left(\frac{\sigma_2}{\sigma_1}X_2(\rho_2 D_2+\sqrt{1-\rho_2^2}Z_2)+\frac{B}{\kappa}\big(\sigma_2 X_2+1\big)-\frac{\sigma_2\rho_2 E}{\kappa}\right)+$$
$$-\frac{\rho_{12}}{2c}\left(\frac{\sigma_1}{\sigma_2}X_2(\rho_1 D_1+\sqrt{1-\rho_1^2}Z_1)+\frac{\sigma_1 B X_2}{\kappa\sigma_2}\Big(\frac{\alpha c_2(2\rho_{12}\sigma_2+\sigma_1)+\sigma_1}{1+\alpha c_2}\Big)+\right.$$
$$\left.+\frac{\rho_{12}B}{\kappa}-\frac{\sigma_1\rho_2 E}{\kappa\sigma_2}\Big(\frac{\alpha c_2(2\rho_{12}\sigma_2+\sigma_1)+\sigma_1}{1+\alpha c_2}\Big)\right).$$

In this example, the weights are functions of normal variables and double stochastic integrals, which are computed via simulation. Table 4 reports the swaption prices in bps with parameters $N=2$, $\alpha=1.5$, $\sigma_1=25\%$, $\sigma_2=15\%$, $c_0=5.28875\%$, $c_1=5.4\%$, $c_2=5.39\%$, $v_0=1$, $\rho_1=-0.75$, $\rho_2=-0.6$, $\kappa=2.3767$, $\theta=0.2143$, $\epsilon_2=25\%$, $\rho_{12}=0.63$.

strikes	*K=3.5%*	*K=4%*	*K=5%*	*K=6%*	*K=7%*	*K=8%*
benchmark	*3.8984*	*2.9221*	*1.2588*	*0.3858*	*0.1019*	*0.0216*
$(0,0)$-*model*	*3.8951*	*2.9053*	*1.2705*	*0.3966*	*0.0942*	*0.0185*
weak Taylor	*3.8990*	*2.9159*	*1.2694*	*0.3791*	*0.1042*	*0.0210*

Table 3: Stochastic volatility swaption values in bps for parameters $\epsilon_1=1$, $\alpha=1.5$, $\sigma_1=25\%$, $\sigma_2=15\%$, $c_0=5.28875\%$, $c_1=5.4\%$, $c_2=5.39\%$, $v_0=1$, $\rho_1=-0.75$, $\rho_2=-0.6$, $\kappa=2.3767$, $\theta=0.2143$, $\epsilon_2=25\%$, $\rho_{12}=0.63$.

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